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On Vector Variational Inequalities in Reflexive Banach Spaces

This paper is dedicated to Professor Franco Giannessi for his 68th birthday

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Abstract. In this paper, we study the solvability for a class of vector variational inequalities in reflexive Banach spaces. By using Brouwer fixed point theorem, we prove the solvability for this class of vector variational inequalities without monotonicity assumption. The solvability results for this class of vector variational inequalities with monotone mappings are also presented by using the KKM-Fan lemma.

Key words: Brouwer fixed point theorem, complete continuity, KKM-Fan lemma, KKM mapping, vector variational inequality.

1. Introduction

Let X and Y be two real Banach spaces, $K \subset X$ be a nonempty, closed and convex set, and $C \subset Y$ a closed, convex and pointed cone with apex at the origin. Recall that C is said to be a closed, convex and pointed cone with apex at the origin iff C is closed and the following conditions hold:

(i) $\lambda C \subset C, \quad \forall \lambda > 0$;
(ii) $C + C \subset P$;
(iii) $C \cap (-C) = \{0\}$.

Given C in Y, we can define relations '$\leqslant_C$' and '$\nleqslant_C$' as follows:

$$x \leqslant_C y \quad \Leftrightarrow \quad y - x \in C$$

and

$$x \nleqslant_C y \quad \Leftrightarrow \quad y - x \notin C.$$

If '$\leqslant_C$' is a partial order, then $(Y, \leqslant_C)$ is called an ordered Banach space ordered by C. Let $L(X, Y)$ denote the space of all continuous linear maps from X into Y and $T: K \to L(X, Y)$ be a nonlinear map. The vector variational inequality (for short, *VVI*) consists in finding $x \in K$, such that:

$$\langle Tx, y-x \rangle \nleq_{\text{int}\, C} 0, \quad \forall y \in K,$$

where $\text{int}\, C$ denotes the interior of C and $a \nleq_{\text{int}\, C} b$ means $b-a \notin \text{int}\, C$.

A *VVI* was first introduced by Giannessi [7] in the setting of finite dimensional Euclidean spaces, later on, a *VVI* was studied and generalized to infinite dimensional spaces by Chen [2, 3], Chen and Craven [4], Chen and Yang [5], Yang [16–18]. In recent years, some generalizations of a *VVI* have been further studied by many authors (see, for instance, [8–15, 19, 20]).

The main purpose of this paper is to present some solvability results for a class of *VVIs* in reflexive Banach spaces by imposing some additional conditions. By using Brouwer fixed point theorem, we prove the solvability for this class of *VVIs* without monotonicity assumption. The solvability results for *VVIs* with monotone maps are also presented by using the KKM-Fan lemma.

From now on, unless other specified, we always suppose that K is a nonempty, closed and convex subset of a real Banach space X and Y is a real Banach space ordered by a nonempty, closed, convex and pointed cone C with $\text{int}\, C \neq \emptyset$. Let $L_c(X, Y)$ be the subspace of $L(X, Y)$, which consists in all completely continuous linear maps from X into Y. Recall that a mapping $g: X \to Y$ is said to be completely continuous iff the weak convergence of x_n to x in X implies that the strong convergence of $g(x_n)$ to $g(x)$ in Y.

For our results, we need the following concepts.

DEFINITION 1.1. A map $A: K \to L(X, Y)$ is said to be monotone iff

$$\langle Ax - Ay, x-y \rangle \geqslant_C 0, \quad \forall x, y \in K,$$

where $a \geqslant_C b$ means $a - b \in C$.

DEFINITION 1.2. A map $f: K \to Y$ is said to be convex iff

$$f(tx + (1-t)y) \leqslant_C t f(x) + (1-t) f(y), \quad \forall x, y \in K, t \in [0, 1].$$

DEFINITION 1.3. A map $A: K \to L(X, Y)$ is said to be v-hemicontinuous iff for any given $x, y \in K$, the mapping $t \longmapsto \langle A(x + t(y-x)), y-x \rangle$ is continuous at 0^+.

2. Solvability of *VVI* without Monotonicity

In this section, we shall present the solvability of a class of *VVIs* in reflexive Banach spaces by using Brouwer fixed pointed theorem. First we give some lemmas.

LEMMA 2.1. [6]. *Let M be a nonempty, closed and convex subset of a Hausdorff topological space and $G: M \to 2^M$ be a multivalued map. Suppose that for any finite set $\{x_1, \ldots, x_n\} \subset M$, one has* conv $\{x_1, \ldots, x_n\} \subset \bigcup_{i=1}^n G(x_i)$ *(i.e., F is a KKM mapping) and $G(x)$ is closed for each $x \in M$ and compact for some $x \in M$, where conv denotes the convex hull operator. Then $\bigcap_{x \in M} G(x) \neq \emptyset$.*

LEMMA 2.2 [5]. *Let Y be a real Banach space ordered by a closed, convex and pointed cone C with apex at the origin and* $\operatorname{int} C \neq \emptyset$. *Then, for any a, b, $c \in Y$, one has*

(i) *$c \not\leqslant_{\operatorname{int} C} a$ and $a \geqslant_C b$ imply that $c \not\leqslant_{\operatorname{int} C} b$;*
(ii) *$c \not\geqslant_{\operatorname{int} C} a$ and $a \leqslant_C b$ imply that $b \not\leqslant_{\operatorname{int} C} c$.*

LEMMA 2.3 [1]. *Let B be a nonempty, compact and convex subset of a finite dimensional space and $F: B \to B$ be a continuous map. Then there exists $x \in B$ such that $F(x) = x$.*

LEMMA 2.4. *Let X be a real Banach space, $K \subset X$ be a nonempty, bounded, closed and convex set, and Y be a real Banach space ordered by a nonempty, closed, convex and pointed cone C. Then, the following conclusions hold:*

(i) *If $T: K \to L_c(X, Y)$ is completely continuous, then for any given $y \in K$, the map $g_y: K \to Y$ defined by $g_y(x) = \langle Tx, y - x\rangle$ is completely continuous*;
(ii) *If $T: K \to L(X, Y)$ is continuous, then for any given $y \in K$, the map g_y: $K \to Y$ defined by $g_y(x) = \langle Tx, y - x\rangle$ is continuous.*

Proof. (i) For any given $x_0 \in K$ and $\epsilon > 0$, since K is bounded and $T: K \to L_c(X, Y)$ is completely continuous, there exists some weak neighbourhood V_1 of x_0 in X, such that:

$$\|\langle Tx - Tx_0, y - z\rangle\| < \frac{\epsilon}{2}, \quad \forall x \in K \cap V_1, y, z \in K.$$

On the other hand, there exists a weak neighbourhood V_2 of x_0 in X, such that:

$$\|\langle Tx_0, x - x_0\rangle\| < \frac{\epsilon}{2}, \quad \forall x \in V_2.$$

Let $V = V_1 \cap V_2$. It follows that

$$\|\langle Tx, y - x\rangle - \langle Tx_0, y - x_0\rangle\| \leqslant \|\langle Tx_0, x_0 - x\rangle\| + \|\langle Tx - Tx_0, y - x\rangle\| < \epsilon$$

for all $x \in K \cap V$. This prove that $g_y: K \to Y$ is continuous from the weak topology of X to the norm topology of Y, i.e., g_y is completely continuous.

Similarly, we can prove conclusion (ii). □

LEMMA 2.5. *Let K be a nonempty, closed and convex subset of a real Banach space X and Y be a real Banach space ordered by a closed, convex and pointed cone C with apex at the origin and* $\operatorname{int} C \neq \emptyset$. *Let $T: K \to L(X, Y)$ be a v-hemicontinuous and monotone map, and $f: K \to Y$ be a convex map. Then, for any given $x_0 \in K$,*

$$\langle Tx_0, y - x_0\rangle + f(Y) - f(x_0) \nleq_{\operatorname{int} C} 0, \quad \forall y \in K$$

if and only if

$$\langle Ty, y - x_0\rangle + f(y) - f(x_0) \nleq_{\operatorname{int} C} 0, \quad \forall y \in K.$$

Proof. Let $x_0 \in K$ such that

$$\langle Tx_0, y - x_0\rangle + f(y) - f(x_0) \nleq_{\operatorname{int} C} 0, \quad \forall y \in K.$$

Since T is monotone,

$$\langle Ty, y - x_0\rangle + f(y) - f(x_0) \geqslant_C \langle Tx_0, y - x_0\rangle + f(y) - f(x_0), \quad \forall y \in K.$$

By Lemma 2.2,

$$\langle Ty, y - x_0\rangle + f(y) - f(x_0) \nleq_{\operatorname{int} C} 0, \quad \forall y \in K.$$

On the other hand, suppose that

$$\langle Ty, y - x_0\rangle + f(y) - f(x_0) \nleq_{\operatorname{int} C} 0, \quad \forall y \in K.$$

For any given $y \in K$, we know that $y_t = ty + (1-t)x_0 \in K$, $\forall t \in (0,1)$ since K is convex. Replacing y by y_t into the above inequality, one has

$$\langle Ty_t, y_t - x_0\rangle + f(y_t) - f(x_0) \nleq_{\operatorname{int} C} 0.$$

It follows from the convexity of f on K that

$$\begin{aligned}
&\langle Ty_t, y_t - x_0\rangle + f(y_t) - f(x_0) \\
&\quad = \langle T[ty + (1-t)x_0], ty + (1-t)x_0 - x_0\rangle + f(ty + (1-t)x_0) - f(x_0) \\
&\quad \leqslant_C \langle T[x_0 + t(y - x_0)], t(y - x_0)\rangle + tf(y) + (1-t)f(x_0) - f(x_0) \\
&\quad = t[\langle T[x_0 + t(y - x_0)], y - x_0\rangle + f(y) - f(x_0)].
\end{aligned}$$

By Lemma 2.2,

$$\langle T(x_0+t(y-x_0)), y-x_0\rangle + f(y) - f(x_0) \nleqslant_{\text{int}\, C} 0, \quad \forall y \in K.$$

Since T is v-hemicontinuous and $Y\backslash(-\text{int}C)$ is closed, letting $t \to 0^+$ in the above inequality, we have

$$\langle Tx_0, y-x_0\rangle + f(y) - f(x_0) \nleqslant_{\text{int}\, C} 0, \quad \forall y \in K.$$

This completes the proof. □

THEOREM 2.1. *Let K be a nonempty, bounded, closed and convex subset of a real reflexive Banach space X and Y a real Banach space ordered by a closed, convex and pointed cone C with apex at the origin and* $\text{int}\, C \neq \emptyset$. *Suppose that $T: K \to L_c(X, Y)$ is a completely continuous map and $f: K \to Y$ is a completely continuous and convex map. Then, there exists $x \in K$ such that*

$$\langle Tx, y-x\rangle + f(y) - f(x) \nleqslant_{\text{int}\, C} 0, \quad \forall y \in K. \tag{1}$$

Proof. If problem (1) is unsolvable, then for every $x_0 \in K$, there exists some $y \in K$ such that

$$\langle Tx_0, y-x_0\rangle + f(y) - f(x_0) \leqslant_{\text{int}\, C} 0. \tag{2}$$

For every $y \in K$, define the set N_y as follows:

$$N_y = \{x \in K : \langle Tx, y-x\rangle + f(y) - f(x) \leqslant_{\text{int}\, C} 0\}. \tag{3}$$

Since both T and f are completely continuous, it follows from (3) and Lemma 2.5 that N_y is open in K with respect to the weak topology of X for every $y \in K$. By (2), we also know that $\{N_y : y \in K\}$ is an open cover of K with respect to the weak topology of X. The weak compactness of K (since K is bounded, closed and convex) implies that, there exists a finite set $\{y_1, \ldots, y_n\} \subset K$, such that:

$$K = \bigcup_{i=1}^{n} N_{y_i}.$$

Hence there exists a continuous (with respect to the weak topology of X) partition of unity $\{\beta_1, \ldots, \beta_n\}$ subordinated to $\{N_{y_1}, \ldots, N_{y_n}\}$ such that $\beta_j(x) \geqslant 0$, $\forall x \in K$, $j = 1, \ldots, n$, $\sum_{j=1}^{n} \beta_j(x) = 1$, $\forall x \in K$, and $\beta_j(x) = 0$ whenever $x \notin N_{y_j}$, $\beta_j(x) > 0$ whenever $x \in N_{y_j}$.

Let $p: K \to X$ be defined as follows:

$$p(x) = \sum_{j=1}^{n} \beta_j(x) y_j, \quad \forall x \in K. \tag{4}$$

Since β_i is continuous with respect to the weak topology of X for each i, p is continuous with respect to the weak topology of X. Let $S = \text{conv}\{y_1, \dots, y_n\} \subset K$. Then S is a simplex of a finite dimensional space and p maps S into S. By Brouwer fixed point theorem (Lemma 2.3), there exists some $x_0 \in S$ such that $p(x_0) = x_0$.

Define $q: K \to Y$ by

$$q(x) = \langle Tx, x - p(x) \rangle + f(x) - f(p(x)), \quad \forall x \in K. \tag{5}$$

For any given $x \in K$, let $k(x) = \{j : x \in N_{y_j}\} = \{j : \beta_j(x) > 0\}$. Obviously, $k(x) \neq \emptyset$. It follows from (3) and (5) that

$$\begin{aligned} q(x) &= \langle Tx, x - p(x) \rangle + f(x) - f(p(x)) \\ &\geqslant_C \sum_{j=1}^{n} \beta_j(x)[\langle Tx, x - y_j \rangle + f(x) - f(y_j)] \\ &= \sum_{j \in k(x)} \beta_j(x)[\langle Tx, x - y_j \rangle + f(x) - f(y_j)] \\ &\geqslant_{\text{int}\, C} 0 \end{aligned}$$

for all $x \in K$. Hence $q(x) \geqslant_{\text{int}\, C} 0$ for all $x \in K$. However, since $x_0 \in S \subset K$ is a fixed point of p, from (5), one has

$$q(x_0) = \langle Tx_0, x_0 - x_0 \rangle + f(x_0) - f(x_0) = 0 \geqslant_{\text{int}\, c} 0.$$

This is a contradiction. Therefore, there exists $x \in K$ such that

$$\langle Tx, y - x \rangle + f(y) - f(x) \not\leqslant_{\text{int}\, C} 0, \quad \forall y \in K$$

This completes the proof. □

THEOREM 2.2. *Let K be a nonempty, compact and convex subset of a real Banach space X and Y a real Banach space ordered by a closed, convex and pointed cone C with apex at the origin and* $\text{int}\, C \neq \emptyset$. *Suppose that $T: K \to L(X, Y)$ is a continuous map and $f: K \to Y$ is a continuous and convex map. Then, there exists $x \in K$ such that*

$$\langle Tx, y - x \rangle + f(y) - f(x) \not\leqslant_{\text{int}\, C} 0, \quad \forall y \in K.$$

Proof. The proof is similar to the proof of Theorem 2.1 and so is omitted. □

If $X = R^n$, then $L_c(R^n, Y) = L(R^n, Y)$ and complete continuity is equivalent to continuity. By Theorem 2.1, we can obtain the following result:

COROLLARY 2.1. *Let K be a nonempty, bounded, closed and convex subset of R^n and Y a real Banach space ordered by a closed, convex and pointed cone C with apex at the origin and* $\operatorname{int} C \neq \emptyset$*. Suppose that $T: K \to L(R^n, Y)$ is a continuous map and $f: K \to Y$ is a continuous and convex map. Then, there exists $x \in K$ such that*

$$\langle Tx, y - x\rangle + f(y) - f(x) \nleq_{\operatorname{int} C} 0, \quad \forall y \in K.$$

Now we give an example which shows that T is completely continuous, but not monotone, and the corresponding *VVI* has a solution.

EXAMPLE 2.1. Let $X = R$, $K = [-\pi/2, \pi/2]$, $Y = R^2$, and $C = R^2_+$. Suppose that $f \equiv 0$ and T is defined by

$$T(x) = \begin{pmatrix} \sin x \cos x \\ \sin^2 x - x \end{pmatrix}.$$

It is easy to verify that T is continuous, but not monotone on K, and 0 is a solution of the corresponding *VVI*.

3. Solvability of *VVI* with Monotonicity

In this section, we prove some existence results for problem (1) with monotonicity assumption by the well-known KKM-Fan lemma (Lemma 2.1).

THEOREM 3.1. *Let K be a nonempty, bounded, closed and convex subset of a real reflexive Banach space X and Y a real Banach space ordered by a closed, convex and pointed cone C with apex at the origin and* $\operatorname{int} C \neq \emptyset$*. Suppose that $T: K \to L_c(X, Y)$ is a v-hemicontinuous and monotone map, and $f: K \to Y$ is a completely continuous and convex map. Then problem* (1) *is solvable.*

Proof. Define two multivalued maps $F, G: K \to 2^k$ as follows:

$$F(y) = \{x \in K : \langle Tx, y - x\rangle + f(y) - f(x) \nleq_{\operatorname{int} C} 0\}, \quad \forall y \in K$$

and

$$G(y) = \{x \in K : \langle Ty, y - x\rangle + f(y) - f(x) \nleq_{\operatorname{int} C} 0\}, \quad \forall y \in K.$$

Then $F(y)$ and $G(y)$ are nonempty since $y \in G(y) \cap F(y)$. We claim that F is a KKM mapping. If it is not the case, then there exist a finite set $\{y_1, \ldots, y_n\} \subset K$, and $t_i \geqslant 0, i = 1, 2, \ldots, n$ with $\sum_{i=1}^n t_i = 1$ such that:

$$y = \sum_{i=1}^n t_i x_i \notin \bigcup_{i=1}^n F(y_i).$$

Hence,

$$\langle Ty, y_i - y\rangle + f(y_i) - f(y) \leqslant_{\text{int}C} 0, \quad i = 1, 2, \ldots, n.$$

It follows that

$$\begin{aligned} 0 &= \langle Ty, y - y\rangle + f(y) - f(y) \\ &\geqslant_C \sum_{i=1}^n t_i \langle Ty, y - y_i\rangle + f(y) - \sum_{i=1}^n t_i f(y_i) \\ &= \sum_{i=1}^n t_i [\langle Ty, y - y_i\rangle + f(y) - f(y_i)] \\ &\geqslant_{\text{int}C} 0. \end{aligned}$$

It is impossible. So F is a KKM mapping. Further, we can prove that $F(y) \subset G(y)$ for every $y \in K$. Indeed, let $x \in F(y)$, i.e.,

$$\langle Tx, y - x\rangle + f(y) - f(x) \nleqslant_{\text{int}C} 0.$$

Since T is monotone,

$$\langle Ty, y - x\rangle + f(y) - f(x) \geqslant_C \langle Tx, y - x\rangle + f(y) - f(x).$$

By Lemma 2.2,

$$\langle Ty, y - x\rangle + f(y) - f(x) \nleqslant_{\text{int}C} 0.$$

Hence $F(y) \subset G(y)$ for every $y \in K$, and so G is also a KKM mapping. It follows from Lemma 2.5 that

$$\bigcap_{y \in K} F(y) = \bigcap_{y \in K} G(y).$$

Furthermore, we know that $G(y) \subset K$ is closed with respect to the weak topology of X since f is completely continuous and T maps K to $L_c(X, Y)$. Since X is reflexive and $K \subset X$ is nonempty, bounded, closed and convex, we know that K is weakly compact and so $G(Y)$ is weakly compact in K. By Lemma 2.1,

$$\bigcap_{y\in K} F(y) = \bigcap_{y\in K} G(y) \neq \emptyset.$$

This implies that there exists $x \in K$ such that:

$$\langle Tx, y-x\rangle + f(y) - f(x) \nleqslant_{\mathrm{int}C} 0, \quad \forall y \in K.$$

This completes the proof. □

REMARK 3.1. In Theorem 3.1, $L_c(X,Y)$ and the complete continuity of f cannot be replaced by $L(X,Y)$ and continuity of f, respectively. Indeed, we can only prove that for every $y \in K, G(y)$ is closed with respect to norm topology of X without convexity of $G(y)$ if T maps K into $L(X,Y)$ and f is continuous. So $G(y)$ need not be weakly compact.

If the boundedness of K is dropped off, then we have the following theorem under certain coercivity condition:

THEOREM 3.2. *Let K be a nonempty, closed and convex subset of a real reflexive Banach space X with $0 \in K$ and Y a real Banach space ordered by a closed, convex and pointed cone C with apex at the origin and* $\mathrm{int}\, C \neq \emptyset$*. Let $T: K \to L_c(X,Y)$ be a v-hemicontinuous monotone map and $f: K \to Y$ be a completely continuous and convex map. If there exists some $r > 0$ such that*

$$\langle Tz, z\rangle + f(z) - f(0) \geqslant_{\mathrm{int}C} 0, \quad \forall z \in K, \|z\| = r, \tag{6}$$

then problem (1) *is solvable.*

Proof. Let $B_r = \{x \in X : \|x\| \leqslant r\}$. By Theorem 3.1, there exists $x_r \in K \cap B_r$ such that:

$$\langle Tx_r, u - x_r\rangle + f(u) - f(x_r) \nleqslant_{\mathrm{int}C} 0, \quad \forall u \in K \cap B_r. \tag{7}$$

Substituting $u = 0$ into the above inequality, one has

$$\langle Tx_r, x_r\rangle + f(x_r) - f(0) \ngeqslant_{\mathrm{int}C} 0. \tag{8}$$

Combining (6) with (8), we know that $\|x_r\| < r$ for some r, For any $y \in K$, choose $t \in (0,1)$ small enough such that $(1-t)x_r + ty \in K \cap B_r$. Substituting $u = (1-t)x_r + ty$ in (7), one has

$$\langle Tx_r, (1-t)x_r + ty - x_r\rangle + f[(1-t)x_r + ty] - f(x_r) \nleqslant_{\mathrm{int}C} 0.$$

Since f is convex,

$$\begin{aligned}&\langle Tx_r, (1-t)x_r + ty - x_r\rangle + f[(1-t)x_r + ty] - f(x_r)\\&\quad \leqslant_C t\langle Tx_r, y - x_r\rangle + (1-t)f(x_r) + tf(y) - f(x_r)\\&\quad = t[\langle Tx_r, y - x_r\rangle + f(y) - f(x_r)].\end{aligned}$$

By Lemma 2.2,

$$\langle Tx_r, y - x_r\rangle + f(y) - f(x_r) \nleqslant_{\mathrm{int}C} 0, \quad \forall y \in K.$$

This completes the proof. □

By Theorems 3.1 and 3.2, we can obtain the following results:

COROLLARY 3.1. *Let K be a nonempty, bounded, closed and convex subset of R^n and Y a real Banach space ordered by a closed, convex and pointed cone C with apex at the origin and* $\mathrm{int}\, C \neq \emptyset$. *Suppose that $T: K \to L(R^n, Y)$ is a v-hemicontinuous monotone map and $f: K \to Y$ is a continuous and convex map. Then, there exists $x \in K$ such that*

$$\langle Tx, y - x\rangle + f(y) - f(x) \nleqslant_{\mathrm{int}C} 0, \quad \forall y \in K.$$

COROLLARY 3.2. *Let K be a nonempty, closed and convex subset of R^n with $0 \in K$ and Y a real Banach Space ordered by a closed, convex and pointed cone C with apex at the orgin and* $\mathrm{int}\, C \neq \emptyset$. *Let $T: K \to L(R^n, Y)$ be a v-hemicontinuous and monotone map and $f: K \to Y$ be a continuous and convex map. If there exists some $r > 0$ such that*

$$\langle Tz, z\rangle + f(z) - f(0) \geqslant_{\mathrm{int}C} 0, \quad \forall z \in K, ||z|| = r,$$

then there exists $x \in K$ such that

$$\langle Tx, y - x\rangle + f(y) - f(x) \nleqslant_{\mathrm{int}C} 0, \quad \forall y \in K.$$

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References

1. Brouwer, L. (1912), Zur Invarianz des n-dimensional Gebietes, *Math. Ann*. 71, 305–313.
2. Chen, G.Y. (1989), Vector variational inequality and its applications for multiobjective optimization, *Chinese Sci. Bull*. 34, 969–972.
3. Chen, G.Y. (1992), Existence of solutions for a vector variational inequality: an extension of Hartman-Stampacchia theorem, *J. Optim. Theory and Appl.* 74, 445–456.
4. Chen, G.Y. and Craven, B.D. (1990), A vector variational inequality and optimization over an Effiecient Set, *Zeitschrift für Oper. Res*. 3, 1–12.
5. Chen, G.Y. and Yang, X.Q. (1990), The vector complementarity problem and its equivalences with the weak minimal element, *J. Math. Anal. Appl*. 153, 136–158.
6. Fan, K. (1984), Some properties of convex sets to fixed point theorem, *Math. Ann*. 266, 519–537.
7. Giannessi, F. (1980), Theorems of alterative, quadratic progams and complementarity problems. In R.W. Cottle, F. Giannessi and J.L. Lions eds., *Variational Inequalities and Complementarity Problems*, Wiley & Sons, New York.
8. Giannessi, F. (Ed.) (2000), *Vector Variational Inequalities and Vector Equilibria*, Kluwer Academic Publishers, Dordrechet, Holland.
9. Giannessi, F. and Maugeri, A. (1995), *Variational Inequalities and Network Equilibrium Problems*, Plenum Press, New York.
10. Hu, J.Y. (1997), Simultaneous vector variational inequalities and vector implicit complementarity problem, *J. Optim. Theory Appl*. 93, 141–151.
11. Konnov, I.V. and Yao, J.C. (1997), On the generalized vector variational inequality problems, *J. Math. Anal. Appl*. 206, 42–58.
12. Lai, T.C. and Yao, J.C. (1996), Existence results for VVIP, *Appl. Math. Lett*. 9, 17–19.
13. Lee, G.M., Kim, D.S. and Lee, B.S. (1996), Generalized vector variational inequality, *Appl. Math. Lett*. 9, 39–42.
14. Lee, G.M., Lee, B.S. and Chang, S.S. (1996), On vector quasivariational inequalities, *J. Math. Anal. Appl*. 203, 626–638.
15. Siddqi, A.H., Ansari, Q.H. and Khaliq, A. (1995), On vector variational inequalities, *J. Optim. Theory Appl*. 84, 171–180.
16. Yang, X.Q. (1993), Vector variational inequality and its duality, *Nonlinear Analysis: TMA* 95, 729–734.
17. Yang, X.Q. (1993), Vector complementarity and minimal element problems, *J. Optim. Theory Appl*. 77, 483–495.
18. Yang, X.Q. (1997), Vector variational inequality and vector pseudolinear optimization, *J. Optim. Theory Appl.* 95, 729–734.
19. Yang, X.Q. and Goh, C.J. (1997), On vector variational inequality application to vector traffic equilibria, *J. Optim. Theory Appl*. 95, 431–443.
20. Yu, S.J. and Yao, J.C. (1996), On vector variational inequalities, *J. Optim. Theory Appl.* 89, 749–769.